Int. J. Heat Mass Transfer. Vol. 17, pp. 257–265. Pergamon Press 1974. Printed in Great Britain

CONJUGATE CONVECTIVE HEAT TRANSFER PROBLEMS

A. V. Luikov
Heat and Mass Transfer Institute, Byelorussian Academy of Sciences, Minsk, U.S.S.R.

(*Received* 21 *February* 1973)

Abstract—From the analysis of a conjugate problem of convective heat transfer in a laminar incompressible flow around a flat plate of a finite thickness the design formulas are suggested for a local Nusselt number Nu_x

$$(Nu_x/Nu_{x0}) - 1 = CB_x, (0 < Br_x < 1{\cdot}5),$$
$$(Nu_x/Nu_{x0}) - 1 = C_0 - (C/Br_x), \quad (1{\cdot}5 < Br_x < \infty),$$

where Nu_{x0} is the Nusselt number with $Br_x = 0$ (the Nusselt number defined by the ordinary heat transfer equations) and Br_x is the local Brun number (a conjugation number).

NOMENCLATURE

a, thermal diffusivity [m^2/s];
Br_x, local Brun number $\left(Br_x = \dfrac{\lambda_f}{\lambda_s}\dfrac{b}{x}Pr^m Re_x^n\right)$;
erfc z = 1-erf z, additional error function;
$$\mathrm{erf}z = \frac{2}{\sqrt{(\pi)}}\int_0^z \exp(-z^2)\,dz,$$
K, B, dimensionless variables defined by formulas (3.11);
l, plate length [m];
n, normal to isothermal surface [m];
Nu_x, local Nusselt number $\left(Nu_x = \dfrac{\alpha_x x}{\lambda_f}\right)$;
$N(K, B)$, dimensionless function defined by formula (3.14);
N^*, the ratio of local Nusselt number Nu_x, to Nu_{x0} in an ordinary problem ($Br_x = 0$) $N^* = Nu_x/Nu_{x0}$;
Pr, Prandtl number;
q, heat flux density [W/m^2];
Re_x, local Reynolds number $\left(Re_x = \dfrac{v_\infty x}{\nu}\right)$;
T, fluid temperature [°C];
θ, plate temperature [°C];
v_∞, fluid velocity far from the wall [m/s];
v_x and v_y, longitudinal and transverse fluid velocities in a boundary layer [m/s];
x^*, dimensionless coordinate $x(x^* = x/l)$;
x, y, z, Cartesian coordinates [m];
α_w, local heat transfer coefficient [W/m^2 deg];
α_x, mean heat transfer coefficient for a flow over a plate [W/m^2 deg];
δ, thickness of velocity boundary layer [m];
b, plate thickness [m];
ξ, thermal boundary layer thickness to velocity boundary layer thickness ratio ($\xi = \delta_r/\delta$);
$\varphi(K, B)$, dimensionless function defined by formula (3.15).

Subscripts

w, refers to the wall surface in contact with fluid;
b, refers to the wall surface maintained at constant temperature;
f, refers to fluid;
s, refers to solid (wall);
a, refers to fluid flow core ($T_\infty \equiv T_a$: $v_\infty \equiv v_a$);
0, Nusselt number value at the Brun number equal to zero ($Br = 0$).

1. STATEMENT OF PROBLEM

Heat transfer between a solid wall and a fluid flow is usually calculated according to the so-called Newton law of convective heat transfer. From this equation the heat flux density q is directly proportional to a temperature driving force $\Delta T(\Delta T = (T_w - T_\infty)$

$$q = \alpha_w(T_w - T_\infty) = \alpha_w \Delta T. \qquad (1.1)$$

Proportionality factor α_w termed a heat-transfer coefficient is variable along the solid surface, and in unsteady heat-transfer processes it depends on time τ.

Relation (1.1) is only valid and physically meaningful at a constant temperature of the wall (T_w = const). In a flow past a plate or in a tube flow the wall temperature is variable along the fluid flow. The behaviour of the wall temperature along the flow cannot be prescribed as it depends on the flow dynamics and thermal conductivity of the wall.

In case of heat sources present at the wall surface or inside the wall ΔT may be negative ($T_w < T_\infty$) at particular regions. The heat transfer coefficient is then negative, that is inconsistent with its physical significance. If x is used to denote a direction of a liquid flow over a flat plate, then relation (1.1) may be written as

$$q = Nu_x \frac{\lambda_f}{x}(T_w - T_\infty) \tag{1.2}$$

where Nu_x is the local Nusselt number.

The surface temperature of the plate depends not only on the x-coordinate and heat sources in the plate but on thermal conductivity of the plate as well. This fact is verified experimentally. In Fig. 1 reproduced from [1] relation $Nu_x = f(x^*)$ is presented for ceramic and glass plates. The figure shows that the curve $f(x^*)$ has extrema and for a glass plate with $x^* > 5$ the local Nusselt number is negative ($Nu_x < 0$). Thus

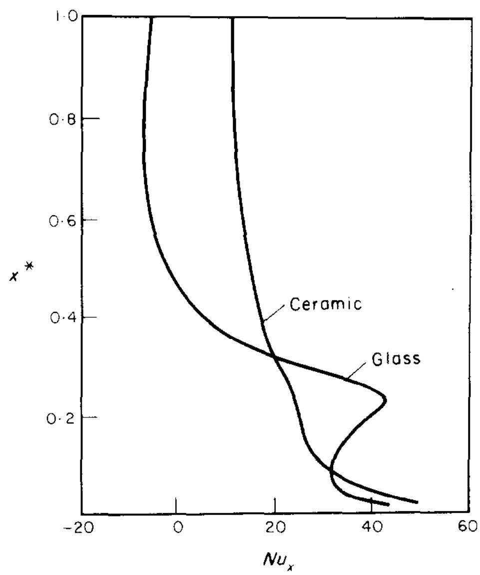

FIG. 1. Local Nusselt numbers along the coordinate x^*—ceramic and glass plates (reproduced from [1]).

for calculation of the Nusselt number Nu_x a simultaneous solution of the problem on heat transfer in a fluid flow over a solid wall and the heat conduction problem inside the solid wall is necessary, i.e. a conjugate problem should be solved. For the solution of a conjugate problem the boundary conditions of the third kind corresponding to convective heat transfer

$$\lambda_f\left(\frac{\partial T}{\partial n}\right)_w + \alpha_w(T_w - T_\infty) = 0 \tag{1.3}$$

should be substituted by the boundary conditions of the fourth kind

$$-\lambda_f\left(\frac{\partial T}{\partial n}\right)_w = -\lambda_s\left(\frac{\partial \theta}{\partial n}\right)_w; \quad T_w = \theta_w \tag{1.4}$$

at the surface.

Relations (1.3) and (1.4) are valid with no heat sources at the wall. Boundary conditions of the fourth kind and conjugate problems of convective heat transfer have originally been formulated in [2] and their solution has first been published in reference [3].

During the last decade a great number of works concerned with conjugate problems have been done and more than one hundred relevant papers have been published [4]. In the papers conjugate problems of heat transfer in a compressible fluid flow over a plate and in a tube flow are attacked. The analytical solutions obtained are extremely complex which is a common disadvantage of all the works. In a great number of works the problems have been solved by numerical methods similar to that described in [1]. The numerical procedure for solution of conjugate problems of convective heat transfer is described in [5]. Nu_x is ordinarily found from the solution of a heat-transfer problem in a boundary layer. For example, for a two-dimensional problem (a flow past a flat plate)

$$Nu_x = \frac{\alpha_x x}{\lambda_f} = -\frac{x}{(T_w - T_\infty)}\left(\frac{\partial T}{\partial y}\right)_w \tag{1.5}$$

where y is a normal to the wall ($y = n$). Here as stated above, a constant wall temperature T_w = const is assumed. In approximate solutions and empirical formulas obtained from experimental data relations of the type

$$Nu_x = APr^m Re_x^n \tag{1.6}$$

are used where A and n are constants.

For example, in a laminar flow over a plate the constants are [4]

$$A = 0{\cdot}332; \quad m = \tfrac{1}{3}; \quad n = \tfrac{1}{2}. \tag{1.7}$$

In conjugate convective heat transfer problems local Nusselt numbers are also found from formula (1.5). The surface temperature of the wall T_w is however found from a solution of the convective heat transfer problem in a boundary layer and the heat conduction problem inside the wall with boundary conditions (1.4). Then relation (1.5) becomes

$$Nu_x = \frac{-x}{[T(0,x) - T_\infty]}\left(\frac{\partial T}{\partial y}\right)_w = \frac{\lambda_s}{\lambda_f}\left(\frac{\partial \theta}{\partial y}\right)_w \frac{x}{[T(0,x) - T_\infty]}. \quad (1.8)$$

The aim of the present paper is an attempt to get approximate solutions of conjugate convective heat transfer problems which would be useful for engineering practice.

2. CONJUGATION NUMBER

As has been mentioned above, a correct physical statement of a convective heat transfer problem between a wall and the surrounding fluid is a conjugate form. If thermal conductivity of the wall is however large compared with that of the fluid ($\lambda_s \gg \lambda_f$), the wall temperature T_w will be slightly variable both across and along the wall (along the y- and x-coordinates, respectively). Then boundary conditions of the third kind (1.3) may be used, and the problem can be solved by a conventional method from relations (1.5) and (1.6).

The choice whether the problem should be solved in a conjugate form or ordinary method is to be used depends not only on the ratio of the fluid-to-solid thermal conductivities λ_f/λ_s but also on the fluid flow dynamics.

A conjugation number named the Brun number is thus a criterion of validity of an ordinary statement of a problem.

Now use will be made of a generalized variable method [6]. It follows from boundary condition of the fourth kind (1.4) that

$$\frac{\Delta\theta}{\Delta T} = \frac{\lambda_f}{\lambda_s}\frac{b}{\delta'_T(x)} \quad (2.1)$$

where $\Delta\theta$ is the temperature drop across the plate, δ'_T is the conventional thickness of a boundary layer. Its numerical value is equal to a section of the straight line $\vartheta_\infty = \text{const}$ ($\vartheta_\infty = T_\infty - T_w$) cut by a tangential line to the temperature distribution curve $T(y)$ at $y = 0$.

The quantity $\Delta\theta/\Delta T$ should not exceed 0·05 within 5 per cent. Then a temperature drop over a plate may be neglected and the heat-transfer problem may be solved by an ordinary (traditional) method.

The quantity $\delta'_T(x)$ is known [4] to be equal to

$$\delta'_T(x) = x Nu_x^{-1}. \quad (2.2)$$

On the other hand the local Nusselt number Nu_x depends on the local Reynolds number Re_x and Prandtl number Pr defined by relation (1.6). Thus

$$\frac{x}{\delta'_T(x)} = Nu_x = A Pr^m Re_x^n, \quad (2.3)$$

and formula (2.1) may be written as

$$\frac{\Delta\theta}{\Delta T} = A\frac{\lambda_f}{\lambda_s}\frac{b}{x} Pr^m Re_x^n. \quad (2.4)$$

The quantity proportional to $\Delta\theta/\Delta T$ but A times smaller

$$Br_x = \frac{\lambda_f}{\lambda_s}\frac{b}{x} Pr^m Re_x^n \quad (2.5)$$

is adopted as the Brun number Br_x.

Formula (2.5) shows that Br_x depends on the thermal conductivity ratio λ_f/λ_s and on Pr and Re_x.

Physically the local Brun number is proportional to the ratio of thermal resistances of the wall and boundary layer over the length x.

For solutions of conjugate problems used for engineering practice a simple relation similar to formula (1.6) of the type

$$Nu_x = A Br^m Re_x^n f(Br_x) \quad (2.6)$$

may be recommended.

To find the form of the function $f(Br_x)$ use will be made of the analytical solution of the transpiration cooling of a plate presented in [7].

3. CONJUGATE PROBLEM SOLUTION BY DIFFERENTIAL HEAT TRANSFER EQUATION

Take a plate with the length l exceeding considerably the thickness b($b/l \ll 1$). The plate is placed in a heated air flow with the constant temperature $T_a = T_\infty = \text{const}$.

For an incompressible fluid a differential heat-conduction equation in a laminar boundary layer flow is of the form

$$v_x\frac{\partial T(x,y)}{\partial x} + v_y\frac{\partial T(x,y)}{\partial y} = a\frac{\partial^2 T(x,y)}{\partial y^2}. \quad (3.1)$$

Boundary conditions may be written in accordance with Fig. 2 as

at

$$x = 0; T(0,y) = T_\infty \quad (3.2)$$

at

$$y = -b; \theta(-b, x = T_b = \text{const} \quad (3.3)$$

at

$$y = 0; -\lambda_f\frac{\partial T(x,0)}{\partial y} = -\lambda_s\frac{\partial \theta(x,0)}{\partial y} \quad (3.4)$$

$$T(x,0) = \theta(x,0) \quad (3.5)$$

at

$$y = \infty\ ; T(x,\infty) = T_\infty = \text{const}. \quad (3.6)$$

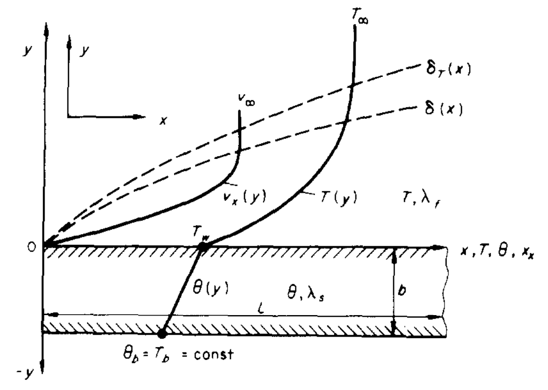

FIG. 2. Schematic diagram of the calculation procedure.

Thus one of the plate surfaces is maintained at a constant temperature, the other is heated by a fluid flow at a temperature T_a (T_a = const). At the latter surface ($y = 0$) a boundary condition of the fourth kind (conjugation condition) is substituted for a boundary condition of the third kind. Thus the temperature of this wall is variable along the co-ordinate x as $T(0, x)$. With an ordinary (traditional) statement of the problem a constant temperature of the surface is assumed $T(0, x) = T_w$ = const.

For small values $b(b/l \ll 1)$ a linear temperature distribution $\theta(y)$ in a plate may be assumed

$$\text{at} \quad y = 0 \quad -\frac{\partial T}{\partial y} + \frac{\lambda_s}{b\lambda}[T(x, 0) - T_b] = 0. \tag{3.7}$$

To analyse the solution of equation (3.1) the following assumptions may be adopted†

$$v_x = \bar{v}_x = \text{const}, \tag{3.8}$$

$$v_y = \bar{v}_y = \text{const} \tag{3.9}$$

where $\bar{v}_x$ and $\bar{v}_y$ are *mean* longitudinal and transverse velocities in a boundary layer, respectively.

The solution of equation (3.1) with (3.8) and (3.9) and boundary conditions (3.2)–(3.6) is published in [4] and [7]

$$\frac{T(x, y) - T_b}{T_\infty - T_b} = \frac{K - \frac{1}{2}B}{K - B} \exp\left[(K^2 - BK) + \lambda^* \frac{y}{b}\right] \times \operatorname{erfc}\left(K - \tfrac{1}{2}B + \tfrac{1}{2}\lambda^* \frac{y}{Kb}\right) + \tfrac{1}{2}\operatorname{erfc}\left(\tfrac{1}{2}B - \frac{1}{2K}\lambda^* \frac{y}{b}\right) - \tfrac{1}{2}\frac{\exp[(B/K)\lambda^*(y/b)]}{1 - (B/K)} \times \operatorname{erfc}\left(\tfrac{1}{2}B + \frac{1}{2K}\lambda^* \frac{y}{b}\right) \tag{3.10}$$

† The error introduced by this assumption will be discussed later.

where $\lambda^* = \lambda_s/\lambda_f$ is the dimensionless thermal conductivity

$$K = \frac{\lambda_s}{\lambda_f}\frac{x}{b}(Pr)^{-\frac{1}{2}}(\overline{Re}_x)^{-\frac{1}{2}}, \quad B = \frac{\bar{v}_y}{\bar{v}_x}\sqrt{(Pr\,\overline{Re}_x)} \tag{3.11}$$

where $\overline{Re}_x$ is the averaged local Reynolds number

$$\overline{Re}_x = \frac{\bar{v}_x x}{\nu}. \tag{3.12}$$

Solution (3.10) implies that the temperature of the upper surface of the plate ($T(0, x) = \theta(0, x)$ decreases as x increases and at $x \to \infty$, the temperature approaches that of the lower surface $T_b[T(0, \infty) = T_b]$.

From the solution of (3.10) using formula (1.8), the following formula for a Nusselt number Nu_x has been obtained [4]

$$Nu_x = \frac{1}{\sqrt{\pi}} Pr^{0{\cdot}5}(\overline{Re}_x)^{0{\cdot}5} N(K, B), \tag{3.13}$$

$$N(K, B) = \frac{\varphi(K, B) - \frac{1}{2}\sqrt{\pi}\, B \operatorname{erfc} B}{\left(1 - \frac{B}{K}\right) - \frac{1}{K\sqrt{\pi}}\varphi(K, B) + \frac{1}{2}\frac{B}{K}\operatorname{erfc}\frac{1}{2}B} \tag{3.14}$$

$$\varphi(K, B) = \left(1 - \tfrac{1}{2}\frac{B}{K}\right)(\pi)K \exp(K^2 - BK) \times \operatorname{erfc}(K - \tfrac{1}{2}B). \tag{3.15}$$

Comparison of formulas (2.5) and (3.11) reveals that dimensionless quantity K is inversely proportional to the local Brun number since the flow over a plate is laminar ($n = 0{\cdot}5$).

To find the relation between $\bar{v}_x$, $\bar{v}_y$ and v_∞ use will be made of the Blasius solution for a flat plate in an incompressible fluid flow.

In this solution an integral mean of (v_x/v_∞) is defined by the function $f(\xi)$ (see formula (3.1.19) in [4]. With $\xi \geqslant 5$ the velocity v_x practically (within 1 per cent) equals v_∞. Then $[f(5)/5]$ is equal to 0·6566 (~0·66). The transverse velocity distribution depends on the quantity $(v_y/v_\infty)Re_x^{0{\cdot}5}$ which beginning from $\xi \geqslant 5{\cdot}2$ is actually equal to a limit value 0·8604 (see Fig. 3.3 in [4]). The complicated distribution may be replaced by a linear one (see formulas (3.1.25) in [4]). Then a mean value $\bar{v}_y$ will be $0{\cdot}43\, v_\infty Re_x^{-0{\cdot}5}$

$$\bar{v}_x = 0{\cdot}66\, v_\infty, \quad \frac{\bar{v}_y}{v_\infty} Re_x^{0{\cdot}5} = 0{\cdot}43. \tag{3.16}$$

Using formulas (3.16) we obtain

$$\frac{1}{K} = 0{\cdot}81\frac{Pr^{\frac{1}{2}}}{Pr^{\frac{1}{3}}}Br_x, \quad (3.17)$$

$$B = \frac{0{\cdot}43}{0{\cdot}81}\sqrt{(Pr)} = 0{\cdot}53\,Pr^{\frac{1}{2}} \quad (3.18)$$

which implies that B is a function of the Prandtl number (for air $Pr = 0{\cdot}7$, $B = 0{\cdot}44$).

Relations (3.13) are rewritten as

$$Nu_x = 0{\cdot}81\left(\frac{Pr}{\pi}\right)^{0\cdot5}\sqrt{(Re_x)}N(K, B). \quad (3.19)$$

If the plate thickness is close to zero, or Br_x vanishes ($Br_x \to 0, K \to \infty$) then from solution (3.19) a solution of an ordinary heat transfer problem is obtainable.

Now analysis will be made of formula (3.13) which is written with relation (3.16) as

$$Nu_{x0} = 0{\cdot}81\left(\frac{Pr}{\pi}\right)^{0\cdot5}\sqrt{(Re_x)}N(\infty, B) \quad (3.20)$$

where Nu_{x0} is the value of the Nusselt number when $K \to \infty$ or $Br_x \to 0$. As $K \to \infty$, the function $\varphi(\infty, B)$ is

$$\varphi(x, B) = \exp\left(-\tfrac{1}{4}B^2\right) \quad (3.21)$$

and the function $N(\infty, B)$

$$N(\infty, B) = \exp\left(-\tfrac{1}{4}B^2\right) - \tfrac{1}{2}\sqrt{(\pi)}B\,\mathrm{erfc}\,\tfrac{1}{2}B. \quad (3.22)$$

For air ($Pe = 0{\cdot}7$) $B = 0{\cdot}44$ and $N(\infty, B)$ is 0·659 from formula (3.22) [$N(\infty, B) = 0{\cdot}659 \approx 0{\cdot}66$]. Thus

$$Nu_x = 0{\cdot}81 \times 0{\cdot}472 \times 0{\cdot}66\sqrt{(Re_x)} = 0{\cdot}252\sqrt{(Re_x)}. \quad (3.23)$$

Comparison of formula (3.23) with (1.6) with account for (1.7) yields

$$Nu_x = 0{\cdot}332Pr^{\frac{1}{3}}\sqrt{(Re_x)} = 0{\cdot}294\sqrt{(Re_x)}. \quad (3.24)$$

The only difference between formulas (3.23) and (3.24) is the different factor A ($A = 0{\cdot}252$ and $A = 0{\cdot}294$). The difference between the two factors A is about 14 per cent. Formula (3.18) implies that if $Pr \to 0$, $B \to 0$, that is the extreme value $B = 0$ corresponds to $Pr \to 0$.

Moreover the above analysis has revealed that solution of differential equation (3.1) with assumptions (3.8) and (3.9) gives a correct relation between Nu_x and Re_x. These assumptions only affect the value of factor A in formula (1.6). Relation (3.19) may therefore be used for analysis of a conjugate problem solution. If N^* ($N^* = Nu_x/Nu_{x0}$) is used to denote the ratio of local Nusselt numbers Nu_x and Nu_{x0}, it may then be written

$$N^* - 1 = \frac{Nu_x - Nu_{x0}}{Nu_{x0}} = \frac{N(K, B) - N(\infty, B)}{N(\infty, B)}. \quad (3.25)$$

If $K \to \infty (Br_x \to 0)$, then $N^* = 1$ or $Nu_x = Nu_{x0}$, thus an ordinary solution of a heat transfer problem is obtained. The function $N(K, B)$ has been calculated for K between 0·1 and 10 for the range of B from 0 to 10·0 (see Table 3·3 in [4]). If $N^* - 1) = f(K)$ is plotted with the data of the table, then for large K ($1{\cdot}5 < K < 10$) or small Br ($0 < Br_x < 0{\cdot}7$), a family of straight lines is obtained which cross the coordinate origin (see Fig. 3). Thus it may be written

$$N^* - 1 = CBr_x \quad (3.26)$$

where constant C depends on B value. In the same figure $C = f(B)$ is plotted. The plot shows that for $B = 0{\cdot}44$ $C = 0{\cdot}56$. Thus for air ($Pr = 0{\cdot}7$)

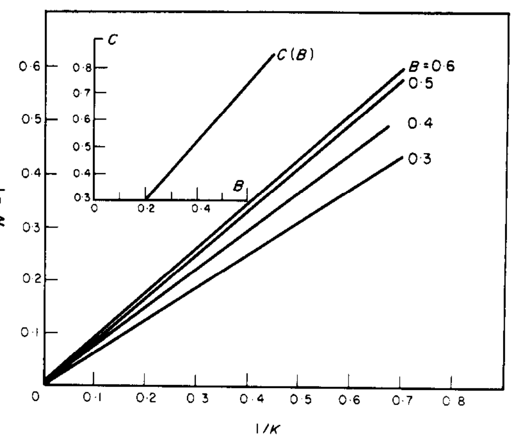

FIG. 3. Plot of $(N^* - 1)$ vs $1/K$ for different values of B.

$$N^* - 1 = 0{\cdot}56\,Br_x. \quad (3.27)$$

It is of interest to compare formula (3.27) with a similar expression resulting from an exact solution of a conjugate problem. In work [4] plots of $B/B^* = f(\xi, \kappa)$ are presented (see Fig. 4.7) for three values of $\varkappa$ (0·639, 2·64, 3·95) where $B/B^* \equiv N^*$, $\xi = x/b$. The dimensionless $\varkappa$ is

$$\varkappa = \frac{1}{2}\frac{\lambda_f}{\lambda_s}\left(\frac{v_\infty b}{\nu}\right)^{0\cdot5} = 0{\cdot}56\sqrt{(\xi)}Br_x. \quad (3.28)$$

If the graphs presented in Fig. 4.7 of [4] are replotted as $(N^* - 1) = f(Br_x)$, a straight line will be obtained crossing the coordinate origin (see Fig. 4)

$$N^* - 1 = 0{\cdot}48 Br_x; \qquad 0 < Br_x < 1{\cdot}5 \qquad (3.29)$$

that is very similar to formula (3.27). For small

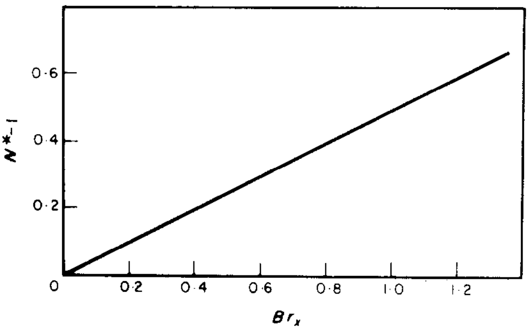

FIG. 4. Plot of $(N^* - 1)$ vs local number Br_x.

values of K ($Br_x > 1$) use of formula (3.25) yields the plots of $(N^* - 1) = f(K)$ shown in Fig. 5. For K within ($0{\cdot}1 < K < 1{\cdot}0$), the plots show a curve family which cuts the sections on the ordinate with their lengths increasing with B (see Fig. 5). The

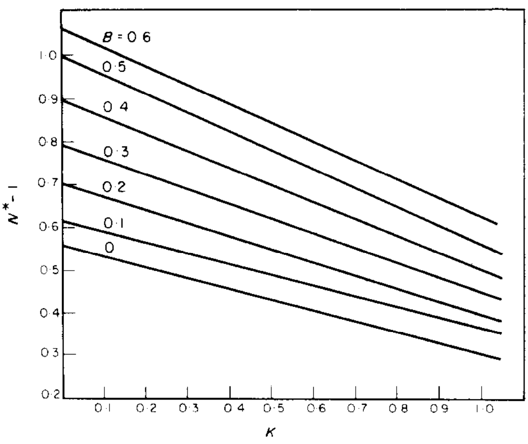

FIG. 5. Plot of $(N^* - 1)$ vs K for different values of B between 0·1 and 0·6.

slope of the straight lines also increases with B. It may approximately be written

$$N^* - 1 = C_0 - C_1' K = C_0 - C_1 \frac{1}{Br_x}, \qquad (3.30)$$

the constants C_0, C_1', C_1 depend on B. For $B = 0{\cdot}44$, $C_0 = 0{\cdot}93$, $C_1' = 0{\cdot}38$ and $C_1 = 0{\cdot}50$

$$Nu^* - 1 = 0{\cdot}93 - 0{\cdot}50 \frac{1}{Br_x}; \ (2 < Br_x < 20). \qquad (3.31)$$

In Fig. 6 $(N^* - 1) = f(1/Br_x)$ is plotted based on

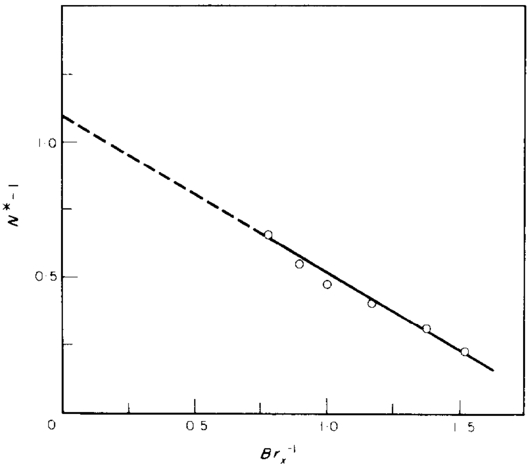

FIG. 6. Plot of $(N^* - 1)$ vs Br_x^{-1}.

the data taken from Fig. 4.7 of [4]. This graph shows that approximate equation may be written as

$$N^* - 1 = 1{\cdot}1 - 0{\cdot}58 \frac{1}{Br_x}; \ (0{\cdot}7 < Br_x < 2). \qquad (3.32)$$

Formula (3.32) is similar to (3.31).

4. CONJUGATE SOLUTION USING BOUNDARY-LAYER EQUATION

The integral heat transfer equation of a boundary layer reads

$$\frac{\partial}{\partial x} \int_0^{\delta_T} (T_\infty - T) v_x(y) \, dy = a \left(\frac{\partial T}{\partial y} \right)_{y=0}. \qquad (4.1)$$

In approximate solutions for a laminar boundary layer the velocity distribution $v_x(y)$ is of the form

$$\frac{v_x}{v_\infty} = \frac{3y}{2\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right). \qquad (4.2)$$

For a temperature distribution in the problem of interest it may be assumed that

$$\vartheta_1 = T - T_b = a_1 + b_1 y + C_1 y^2 + d_1 y^3, \qquad (4.3)$$

$$\vartheta_2 = \theta - \theta_b = a_2 + b_2 y, \qquad (4.4)$$

where $\theta_b = T_b = \text{const}$.

Boundary conditions may be written as follows at $y = \delta_T$

$$\vartheta_1 = \vartheta_\infty = (T_\infty - T_b); \quad \frac{\partial \vartheta}{\partial y} = 0 \qquad (4.5)$$

at $y = 0$

$$\vartheta_1 = \vartheta_2 = \vartheta_w; \quad \lambda_f \frac{\partial \vartheta_1}{\partial y} = \lambda_s \frac{\partial \vartheta_2}{\partial y} \qquad (4.6)$$

at $y = -b$

$$\vartheta_2 = 0. \quad (4.7)$$

Using (4.5)–(4.7) yields

$$\vartheta_1 = \vartheta_w + \tfrac{3}{2}\frac{\vartheta_\infty - \vartheta_w}{\delta_T} y - \tfrac{1}{2}\frac{\vartheta_\infty - \vartheta_w}{\delta_T^3} y^3, \quad (4.8)$$

$$\vartheta_2 = \vartheta_w + \frac{\vartheta_w}{b} y \quad (4.9)$$

where ϑ_w is the boundary temperature variable along x

$$\vartheta_w = \frac{\vartheta_0 z}{1 + z} \quad (4.10)$$

$$z = \tfrac{3}{2}(\lambda_f/\lambda_s)(b/\delta_T). \quad (4.11)$$

Substitution of formulas (4.2) and (4.8) into equation (4.1) gives

$$\frac{d}{dx}\int_0^{\delta_T} V_\infty\left[1 - \tfrac{3}{4}\frac{y}{\delta_T} + \tfrac{1}{2}\left(\frac{y}{\delta_T}\right)^3\right](\vartheta_\infty - \vartheta_w) \times \left[\tfrac{3}{2}\frac{y}{\delta} - \tfrac{1}{2}\left(\frac{y}{\delta}\right)^3\right]dy = a\left(\frac{d\vartheta_1}{\partial y}\right)_{y=0}, \quad (4.12)$$

δ_T is taken as the upper integration limit as at $y > \delta_T$. $\vartheta = \vartheta_\infty \cdot \delta_T/\delta$ is denoted by $\xi(\xi = \delta_T/\delta)$. For gases $\xi > 1$ but differs from unity very slightly. In the subsequent calculation $\xi > 1$ is assumed. This assumption however gives a very slight error [9]. With account for relation (4.10) we have

$$\frac{3}{20}\vartheta v_\infty \frac{d}{dx}\left\{\frac{1}{(1+z)}\left[\frac{\xi^2}{\delta z^2} - \frac{1}{14}\frac{\xi^4}{\delta^3 z^4}\right]\right\} = \tfrac{3}{2}a\frac{\vartheta_\infty z}{\gamma(1+z)}, \quad (4.13)$$

where $\gamma = z\delta_T$

The first term in the square brackets of formula (4.13) is larger by a factor of 14 than the second one. Hence

$$\frac{d}{dx}\left[\frac{1}{(1+z)z^2\delta}\right] = \frac{10az}{\gamma^3 v_\infty(1+z)} \quad (4.14)$$

with $z \to 0$ the equation becomes

$$\frac{d}{dx}\left(\frac{1}{z^2\delta}\right) = \frac{10az}{\gamma^3 v_\infty},$$

hence

$$\xi = \frac{\delta_T}{\delta} = \sqrt[3]{\left(\frac{13}{14}\right)}(\sqrt[3]{Pr})^{-1}, \quad (4.15)$$

that is exactly the same as the solution presented in [9].

The local Nusselt number Nu_x is defined by formula (4.8)

$$Nu_x = \frac{x}{(\vartheta_\infty - \vartheta_w)}\left(\frac{d\vartheta_1}{dy}\right)_w = \tfrac{3}{2}\frac{x}{\delta_T}. \quad (4.16)$$

Using the known approximate relation

$$\frac{\delta}{x} = \sqrt{\left(\frac{280\nu}{13v_\infty x}\right)} = \frac{4{\cdot}64}{\sqrt{Re_x}}, \quad (4.17)$$

a formula for Nu_{x0} in accordance with (4.16) at $z \to 0$ is obtained as

$$Nu_{x0} = 0{\cdot}332\,(Pr)^{1/3}(Re_x)^{1/2}, \quad (4.18)$$

which is well known in the heat-transfer theory.

Asymptotic solutions of equation (4.14) may be obtained at large $z(z \to \infty)$ and small $z(z \to 0)$.

The approximate solution of equation (4.14) yields

$$Nu_x = \tfrac{3}{2}\frac{x}{\delta}Pr^{1/3}\beta[1 + p(x)] \quad (4.19)$$

where β is a constant factor, $p(x)$ is some small function of x so that $\rho^2 x0$. As $z \to 0$ (small Br_x)

$$N^* - 1 = \frac{Nu_x - Nu_{x0}}{Nu_{x0}} \approx 0{\cdot}33\,Br_x \quad (4.20)$$

that is very similar to formulas (3.27)–(3.29). They only differ by a value of C.

With large Br_x ($x \to \infty$ or $x \to 0$) a qualitative relation between N^* and Br_x is only obtainable as with $Br_x \to \infty$ formula (4.8) fails to describe a temperature distribution

$$N^* - 1 = 0{\cdot}66 - \frac{0{\cdot}34}{Br_x} \quad (4.21)$$

that with respect to $N^* = f(Br_x)$ is the same as formula (3.30).

5. DESIGN FORMULAS

The analysis has been presented above of approximate solutions for the Nusselt number in a laminar flow around a flat plate with thickness b and appreciable length 1 ($b/l \ll 1$). The solutions have been obtained from differential and integral relations of a boundary layer. These solutions have been compared with exact solutions for Br_x in the range ($0 < Br_x < 1{\cdot}5$). The comparison has revealed good agreement.

It is demonstrated that if heat flux to the plate is accounted for, the resultant Nusselt number is larger since $N^* > 1$ ($Nu_x > Nu_{x0}$). This increase may be explained as follows. The surface temperature of the wall $T(0, x)$ decreases along x from T_∞ at $x = 0$ to T_b at $x = \infty$, the temperature driving force $\Delta T(T_a - T_w)$ thus increasing along x. It is known from the heat transfer theory that if the temperature difference ΔT increases in the same direction as the fluid flow, the heat transfer coefficient also increases in the same direction x. The behaviour and the value of $T(0, x)$ depends on the value of the Brun number, which thus characterizes nonuniformity of the plate temperature that directly affects heat transfer.

At small Brun numbers ($Br_x < 1{\cdot}5$) formula (3.26) may be used, at large Brun numbers ($Br_x > 1{\cdot}5$) use is to be made of formula (3.30).

For engineering calculations formulas (3.26) and (3.30) may be substituted by a single formula

$$N^* - 1 = CBr_x^p \qquad Br_x < 20 \qquad (4.22)$$

where exponent p depends on Br_x value and for $Br_x > 1{\cdot}5$, $p = 1$. Constant C depends on Pr. At $Br_x > 1{\cdot}5$ the exponent p and constant C depend on the range of Br_x and Pr.

Formula (3.26) may be used to estimate the minimum Brun number below which the problem may not be solved as a conjugate one. If $C \simeq 0{\cdot}5$ is assumed, the Brun number may be less than 0·1 ($Br_x \leqslant 0{\cdot}1$), the error being 5 per cent. When $Br > 0{\cdot}1$ the problem on a flow around a plate should be solved as a conjugate one.

Estimation of the larger value of Br_x which may be encountered in practice is of interest.

In a laminar water flow over a steel plate ($\lambda_s = 0{\cdot}46$ W/m deg, $\lambda_f = 0{\cdot}68$ W/m deg, $Pr = 1{\cdot}75$) with $Re_b = 10^6$ and ratio $b/l \approx 0{\cdot}05$ the mean Brun number Br_l will be about 1·8 that is appreciably larger than 0·1. In a laminar air flow over a steel plate ($\lambda_f = 0{\cdot}028$ W/m deg, $Pr = 0{\cdot}7$) with the same Re_l and b/l $Br_l = 0{\cdot}05$ which implies that conjugation may be neglected and the problem should be solved by the ordinary method using formula (1.6). In case of a glass plate however ($\lambda_s = 0{\cdot}096$ W/m deg) in an air flow with the same parameters ($Rr_l = 10^6$, $b/l = 0{\cdot}05$), $Br_l = 2{\cdot}6$. Thus the problem should be solved as a conjugate one. The same situation is in the case of a metal plate of titanium alloy ($\lambda_s = 15{\cdot}6$ W/m deg) in an air flow ($Re_l = 10^6$, $Pr = 0{\cdot}7$) wherein $Br_l = 0{\cdot}16$, i.e. more than 0·1).

Formula (4.22) implies that with $Br_l = 0{\cdot}5$ Nu_l increases by 25 per cent compared to Nu_l ($C \approx 0{\cdot}5$, $N^* = 1{\cdot}25$) and with $Br_l = 1$ Nu_l and consequently the heat transfer coefficient will increase by 50 per cent compared to their values calculated by the classical heat transfer equations. To conclude the paper it should be pointed out that in case of a turbulent flow past a plate conjugation is of greater effect on heat transfer since according to formula (2.1)–2.5) the Brun number is directly proportional to the Nusselt number.

REFERENCES

1. A. E. Zinnes, The coupling of conduction with laminar natural convection from a vertical plate with arbitrary surface heating, *Trans. Am. Soc. Mech. Engrs* **C**, w3 528 (1970).
2. A. V. Luikov, *Analytical Heat Diffusion Theory*. Academic Press (1968).
3. A. V. Luikov and T. L. Perelman, Aspects of unsteady heat transfer between a body and a flow around it, *Heat and Mass Transfer*, Vol. 6, p. 63 (1965).
4. A. V. Luikov, *Heat and Mass Transfer*, Handbook. Energiya, Moscow (1972).
5. A. V. Luikov, T. L. Perelman and V. B. Ryvkin, On determination of heat transfer coefficient in simultaneous conductive and convective heat transfer processes, 3rd Int. Heat Transfer Conference, Chicago (1966).
6. A. A. Gukhman, *Introduction to the Similarity Theory*. Vysshaya Shkola, Moscow (1963).
7. A. V. Luikov, *Heat and Mass Transfer in Capillary Porous Bodies*. Pergamon, Oxford (1966).
8. A. V. Luikov, V. A. Aleksashenko and A. A. Aleksashenko, Analytical methods of solution of conjugated problem in convective heat transfer, *Int. J. Heat Mass Transfer* **14**, 1047 (1971).
9. E. R. G. Eckert and S. M. Drake, *Heat and Mass Transfer*. McGraw-Hill (1959).

PROBLÈME CONJUGUE DE TRANSFERT THERMIQUE PAR CONVECTION

Résumé—On analyse un problème conjugué de transfert thermique convectif dans un écoulement laminaire incompressible autour d'une plaque plane d'épaisseur finie. Pour le nombre de Nusselt local Nu_x, on propose les formules suivantes:

$$(Nu_x/Nu_{x0}) - 1 = CBr_x \qquad (0 < Br_x < 1{,}5)$$

$$(Nu_x/Nu_{x0}) - 1 = C_0 - (C/Br_x) \; (1{,}5 < Br_x)$$

où Nu_{x0} est le nombre de Nusselt pour $Br_x = 0$ (le nombre de Nusselt défini par les équations ordinaires) et Br_x est le nombre de Brun local (un nombre lié à la conjugaison).

KONJUGIERTE KONVEKTIVE WÄRMEÜBERTRAGUNGSPROBLEME

Zusammenfassung—Ausgehend von der Analyse eines konjugierten Problems der konvektiven Wärmeübertragung bei laminarer inkompressibler Strömung beidseitig einer ebenen Platte endlicher Dicke sind zwei Gleichungen für die örtliche Nusselt-Zahl Nu_x vorgeschlagen worden.

$$(Nu_x/Nu_{x0}) - 1 = CBr_x \text{ für } 0 < Br_x < 1{,}5$$

$$(Nu_x/Nu_{x0}) - 1 = C_0 - (C/Br_x) \text{ für } 1{,}5 < Br_x$$

Dabei ist Nu_{x0} die Nusselt-Zahl mit $Br_x = 0$ (die Nusselt-Zahl ist durch die Wärmeübergangsgleichung definiert) und Br_x ist die örtliche Brun-Zahl (eine Konjugations-Zahl).

СОПРЯЖЕННЫЕ ЗАДАЧИ КОНВЕКТИВНОГО ТЕПЛООБМЕНА

Аннотация—На основе приближенных решений сопряженной задачи конвективного теплообмена при ламинарном обтекании плоской пластины конечной толщины несжимаемой жидкостью предложены расчетные формулы для лок льного числа Нуссклььта

$$\frac{\mathrm{Nu}_x}{\mathrm{Nu}_{xo}} - 1 = CBr_x (0 < Br_x < 1{,}5)$$

$$\frac{\mathrm{Nu}_x}{\mathrm{Nu}_{xo}} - 1 = C_o - \frac{C_1}{Br_x} (1{,}5 < Br_x < \infty),$$

где Nu_{xo} – значение числа Нуссельта при $Br_x = 0$ (число Нуссклььта, определяемое по общеизвестным формулам теплообмены), Br_x – локальное число число Брюна (критерий сопряженности).